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Derivative of the generalised Riemann zeta function $\zeta(z, q)$ at $z=-1$

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Abstract. Several exact representations (as an integral and as an infinite series) for the partial derivative $\partial\zeta(z, q)/\partial z|_{z=-1}$ of the generalised Riemann zeta function $\zeta(z, q)$ are given.

The generalised zeta function $\zeta(z, q)$ is defined by

$$\zeta(z, q)=\sum_{n=0}^{\infty}(n+q)^{-z} \qquad \operatorname{Re} z>1 \qquad q \neq 0,-1,-2, \ldots . \tag{1}$$

For $q=1$, it reduces to the ordinary zeta function $\zeta(z)$:

$$\zeta(z)=\sum_{n=1}^{\infty} n^{-z}. \tag{2}$$

The only derivatives of $\zeta(z, q)$ which are found in the usual tables are the following [1]:

$$\begin{aligned}
&\frac{\partial}{\partial q} \zeta(z, q)=-z \zeta(z+1, q) \\
&\frac{\partial}{\partial z} \zeta(z, q)|_{z=0}=\log \Gamma(q)-\tfrac{1}{2} \log (2 \pi).
\end{aligned} \tag{3}$$

Also interesting is the asymptotic expansion of $\zeta(z, q)$ for large $|q|$ and $|\arg q|<\pi$

$$\begin{aligned}
\zeta(z, q)=\frac{1}{\Gamma(z)}\bigg(& q^{1-z} \Gamma(z-1)+\tfrac{1}{2} \Gamma(z) q^{-z}+\sum_{n=1}^{m-1} \frac{B_{2n}}{(2n)!} \\
& \times \Gamma(z+2n-1) q^{-2n-z+1}\bigg)+\mathrm{O}(q^{-2m-z-1}).
\end{aligned} \tag{4}$$

For the particular values $z=-m$, $m=0, 1, 2, \ldots$, the function $\zeta(z, q)$ is given by the expression

$$\zeta(-m, q)=-\frac{B_{m+1}(q)}{m+1} \tag{5}$$

where $B_k(q)$ are the Bernoulli polynomials.

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A very useful integral representation of the generalised zeta function $\zeta(z, q)$, valid for Re $q>0$, $z \neq 1$, is given by

$$\zeta(z, q)=\tfrac{1}{2} q^{-z}+\frac{q^{1-z}}{z-1}+2 \int_{0}^{\infty}\left(q^{2}+t^{2}\right)^{-z / 2} \sin \left[z \tan ^{-1}\left(\frac{t}{q}\right)\right] \frac{\mathrm{d} t}{\mathrm{e}^{2 \pi t}-1}. \tag{6}$$

From equation (6) one easily sees that $\zeta(z, q)$ is meromorphic everywhere except for one singularity at $z=1$ (a simple pole with residue 1). A number of different representations of $\zeta(z, q)$ as an integral, a series, or an infinite product can be given [1]. However, the integral representation (6) turns out to be most convenient in order to find the partial derivative

$$\varphi(q) \equiv \frac{\partial}{\partial z} \zeta(z, q)\Big|_{z=-1}. \tag{7}$$

This function turns out to be very important in the effective Lagrangian theory of quark confinement [2].

Let us find the partial derivative of (6) with respect to z:

$$\begin{aligned}
\frac{\partial}{\partial z} \zeta(z, q)= & -\tfrac{1}{2} q^{-z} \log q-\frac{q^{1-z}}{z-1} \log q-\frac{q^{1-z}}{(z-1)^{2}} \\
& +2 \int_{0}^{\infty}\left(q^{2}+t^{2}\right)^{-z / 2} \cos \left[z \tan ^{-1}\left(\frac{t}{q}\right)\right] \tan ^{-1}\left(\frac{t}{q}\right) \frac{\mathrm{d} t}{\mathrm{e}^{2 \pi t}-1} \\
& -\int_{0}^{\infty}\left(q^{2}+t^{2}\right)^{-z / 2} \log \left(q^{2}+t^{2}\right) \sin \left[z \tan ^{-1}\left(\frac{t}{q}\right)\right] \frac{\mathrm{d} t}{\mathrm{e}^{2 \pi t}-1}.
\end{aligned} \tag{8}$$

Putting $z=-1$, we obtain

$$\begin{aligned}
\varphi(q)= & -\tfrac{1}{2} q \log q+\tfrac{1}{2} q^{2} \log q-\tfrac{1}{4} q^{2} \\
& +2 q \int_{0}^{\infty} \tan ^{-1}\left(\frac{t}{q}\right) \frac{\mathrm{d} t}{\mathrm{e}^{2 \pi t}-1}+\int_{0}^{\infty} t \log \left(q^{2}+t^{2}\right) \frac{\mathrm{d} t}{\mathrm{e}^{2 \pi t}-1}
\end{aligned} \tag{9}$$

and making an immediate change of variables in the integrals, we get

$$\begin{aligned}
\varphi(q)= & -\tfrac{1}{2} q \log q-\tfrac{1}{4} q^{2}+\tfrac{1}{2} q^{2} \log q+2 \log q \int_{0}^{\infty} \frac{t\, \mathrm{d} t}{\mathrm{e}^{2 \pi t}-1} \\
& +q^{2} \int_{0}^{\infty} \frac{2 \tan ^{-1} x+x \log \left(1+x^{2}\right)}{\mathrm{e}^{2 \pi q x}-1} \mathrm{d} x.
\end{aligned} \tag{10}$$

The first integral is trivial to calculate:

$$\int_{0}^{\infty} \frac{t\, \mathrm{d} t}{\mathrm{e}^{2 \pi t}-1}=\frac{1}{4 \pi^{2}} \int_{0}^{\infty} \frac{t\, \mathrm{d} t}{\mathrm{e}^{t}-1}=\frac{1}{4 \pi^{2}} \frac{\pi^{2}}{6}=\frac{1}{24} \tag{11}$$

but the second one

$$I(q) \equiv \int_{0}^{\infty} \frac{2 \tan ^{-1} x+x \log \left(1+x^{2}\right)}{\mathrm{e}^{2 \pi q x}-1} \mathrm{d} x \tag{12}$$

cannot be computed analytically. It can be written as

$$I(q)=\sum_{k=1}^{\infty} \int_{0}^{\infty}\left[2 \tan ^{-1} x+x \log \left(1+x^{2}\right)\right] \mathrm{e}^{-2 \pi k q x} \mathrm{d} x \qquad q>0 \tag{13}$$

and, after integrating by parts twice,

$$I(q)=\frac{1}{2\pi^2 q^2}\sum_{k=1}^{\infty}\frac{1}{k^2}+\frac{1}{2\pi^2 q^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\infty}\frac{x\,\mathrm{e}^{-2\pi kqx}}{1+x^2}\,\mathrm{d}x. \tag{14}$$

After two more partial integrations, we get

$$q^2 I(q)=\frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}+\frac{1}{8\pi^4 q^2}\sum_{k=1}^{\infty}\frac{1}{k^4}+\frac{1}{4\pi^4 q^2}\sum_{k=1}^{\infty}\frac{1}{k^4}\int_0^{\infty}\left(\frac{1}{(1+x^2)^2}-\frac{4}{(1+x^2)^3}\right)x\,\mathrm{e}^{-2\pi kqx}\,\mathrm{d}x \tag{15}$$

and after another two, we get

$$q^2 I(q)=\frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}+\frac{1}{8\pi^4 q^2}\sum_{k=1}^{\infty}\frac{1}{k^4}-\frac{3}{16\pi^6 q^4}\sum_{k=1}^{\infty}\frac{1}{k^6}+\frac{3}{4\pi^6 q^4}\sum_{k=1}^{\infty}\frac{1}{k^6}\int_0^{\infty}\left(\frac{1}{(1+x^2)^3}-\frac{12}{(1+x^2)^4}+\frac{16}{(1+x^2)^5}\right)x\,\mathrm{e}^{-2\pi kqx}\,\mathrm{d}x. \tag{16}$$

Now, making use of

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}=1.6449 \qquad \sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\pi^4}{90}=1.0823 \qquad \sum_{k=1}^{\infty}\frac{1}{k^6}=\frac{\pi^6}{945}=1.0173 \tag{17}$$

and substituting these values into equation (10), we obtain

$$\varphi(q)=-\tfrac{1}{2}q\log q-\tfrac{1}{4}q^2+\tfrac{1}{2}q^2\log q+\tfrac{1}{12}\log q+\tfrac{1}{12}+\frac{1}{720q^2}-\frac{1}{5040q^4}+\mathrm{O}\left(\frac{1}{q^6}\right) \qquad q>0. \tag{18}$$

For $q>1$ we find a very strong convergence of these first terms in $1/q$. However, we must not forget that equation (16) is an identity, valid for any $q>0$ (no approximation has been carried out yet). The following bound on the higher order terms in (18) is thus also general:

$$\begin{aligned}\mathrm{O}\left(\frac{1}{q^6}\right)&\equiv\frac{3}{4\pi^6 q^4}\sum_{k=1}^{\infty}\frac{1}{k^6}\int_0^{\infty}\left(\frac{1}{(1+x^2)^3}-\frac{12}{(1+x^2)^4}+\frac{16}{(1+x^2)^5}\right)x\,\mathrm{e}^{-2\pi kqx}\,\mathrm{d}x\\&<\frac{3}{4\pi^6 q^4}\sum_{k=1}^{\infty}\frac{1}{k^6}\int_0^{\infty}\left(\frac{x}{(1+x^2)^3}+\frac{12x}{(1+x^2)^4}+\frac{16x}{(1+x^2)^5}\right)\mathrm{d}x\\&<\frac{1}{560q^4} \qquad q>0.\end{aligned} \tag{19}$$

Moreover, notice that, in equation (18), the term in q^{-4} exceeds the term in q^{-2} only for

$$q^2<\tfrac{1}{7} \tag{20}$$

which is already a rather small value. Even in this case (20), the higher order terms in $1/q$ compensate to give a contribution subject to the bound (19).

Alternatively, an exact expression in series form can be given for $\varphi(q)$. Let us go back to equation (10), with $q^2 I(q)$ given by equation (14). We find

$$\varphi(q) = -\tfrac{1}{2}q\log q - \tfrac{1}{4}q^2 + \tfrac{1}{2}q^2\log q + \tfrac{1}{12}\log q + \tfrac{1}{12} + \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} \times\{\sin(2\pi kq)[\tfrac{1}{2}\pi - \mathrm{Is}(2\pi kq)] + \cos(2\pi kq)\mathrm{Ic}(2\pi kq)\} \qquad (21)$$

where $\mathrm{Is}(t)$ and $\mathrm{Ic}(t)$ are the sine and the cosine integral, respectively

$$\mathrm{Is}(t) = \int_0^t \frac{\sin u}{u}\,\mathrm{d}u \qquad \mathrm{Ic}(t) = \int_t^{+\infty}\frac{\cos u}{u}\,\mathrm{d}u. \qquad (22)$$

A point to be remarked on is that the expression used for $\varphi(q)$ has no meaning at $q = 0$. Therefore, the expansion given by (18) is not the best one for $q \simeq 0$, although—as we have already noticed—it is actually valid for every $q > 0$†.

A different exact expression of (18) as an infinite series expansion can be given. One just has to follow the procedure of partial integration '*ad infinitum*'. The result is

$$\varphi(q) = -\tfrac{1}{2}q\log q - \tfrac{1}{4}q^2 + \tfrac{1}{2}q^2\log q + \tfrac{1}{12}\log q + \tfrac{1}{12} - \sum_{n=1}^{m}\frac{B_{2n+2}}{(2n+2)(2n+1)2n}q^{-2n} + \mathrm{O}(q^{-(2m+2)}) \qquad (23)$$

where the B_n are Bernoulli numbers. This series does not converge (it is asymptotic, for any value of q) and, therefore, the correct treatment of the function $\varphi(q)$ must always follow the path of the first procedure, i.e. one must write expressions (14), (15), (16) or (18) to the desired order in $1/q$ and then find a bound for the remainder, to get in this way a conveniently small error.

References

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† The alternative to (18) for $q \simeq 0$ is presently under investigation.